

# INVERSE NODAL PROBLEM FOR P-LAPLACIAN ENERGY-DEPENDENT STURM-LIOUVILLE EQUATION

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**Abstract:** In this study, inverse nodal problem is solved for  $p$ -Laplacian Schrödinger equation with energy-dependent potential with the Dirichlet conditions. Asymptotic estimates of eigenvalues, nodal points and nodal lengths are given by using Prüfer substitution. Especially, an explicit formula for potential function is given by using nodal lengths. Results are more general than classical  $p$ -Laplacian Sturm Liouville problem. For the proofs, it is used the methods given in the references [18], [23].

**MSC 2000 :** 34A55, 34L20

**Key Words:** Prüfer Substitution, Inverse nodal problem,  $P$ -Laplacian equation

## 1. Introduction

Consider the following eigenvalue problem for  $p$ -Laplacian

$$-\left(u'^{(p-1)}\right)' = (p-1)(\lambda - q(x))u^{(p-1)} \quad (1.1)$$

with the conditions

$$u(0) = u(1) = 0 \quad (1.2)$$

where  $p \in [1, \infty)$  is a constant,  $\lambda$  is the eigenvalue and  $q \in L^2(0, 1)$ . Equation (1.1) is also known as one-dimensional  $p$ -Laplacian eigenvalue equation. Note that when  $p = 2$ , equation (1.1) becomes Sturm-Liouville equation as

$$-u'' + q(x)u = \lambda u$$

and inverse problem was solved for (1.1),(1.2) in the references [1], [2], [3], [6], [7] [18], [22], [23].

The determination of the form of a differential operator from spectral data associated with it has enjoyed close attention from a number of authors in last years. One of the these operators is Sturm - Liouville operator. In the typical formulation of the inverse Sturm-Liouville problem one seeks to recover both  $q(x)$  and constants by giving the eigenvalues with another piece of spectral data. These data can take several forms, leading to many versions of the problem.

Especially, the recent interest is a study by Hald and McLaughlin [10], [20] wherein the given spectral information consists of a set of nodal points of eigenfunctions for the Sturm-Liouville problems. These results were extended to the case of problems with eigenparameter dependent boundary conditions by Browne and Sleeman [4]. On the other hand, Law et al [16], Law and Yang [17], solved the inverse nodal problem of determining the smoothness of the potential function  $q(x)$  of the Sturm-Liouville problem by using nodal data. In the past few years, the inverse nodal problem of Sturm-Liouville problem has been investigated by several authors [5], [15], [25].

When  $q = 0$ , consider the problem

$$-\left(u'^{(p-1)}\right)' = (p-1)\lambda u^{(p-1)}$$

$$u(0) = u(1) = 0.$$

The eigenvalues of this problem were given as [18]

$$\lambda_n = (n\pi_p)^p, n = 1, 2, 3, \dots,$$

where

$$\pi_p = 2 \int_0^1 \frac{dt}{(1-t^p)^{1/p}} = \frac{2\pi}{p \sin\left(\frac{\pi}{p}\right)}$$

and associated eigenfunction is denoted by  $S_p(x)$ .  $S_p(x)$  and  $S'_p(x)$  are periodic functions satisfying the identity

$$[S_p(x)]^p + [S'_p(x)]^p = 1$$

for arbitrary  $x \in \mathbb{R}$ . These functions known as generalized sine and cos functions and for  $p = 2$  become  $\sin x$  and  $\cos x$  [19].

Now, we must present some further properties of  $S_p(x)$  for deriving a more detailed asymptotic formulas. These formulas are crucial in the solution of our problem.

**Lemma 1.1.** [18]

a) For  $S'_p \neq 0$ ,

$$(S'_p)' = - \left| \frac{S_p}{S'_p} \right|^{p-2} \cdot S_p$$

b)

$$\left(S_p S_p'^{(p-1)}\right)' = |S'_p|^p - (p-1)S_p^p = 1 - p|S_p|^p = (1-p) + p|S'_p|^p.$$

According to the Sturm-Liouville Theory, the zero set  $x_n = \{x_j^{(n)}\}_{j=1}^n$  of the eigenfunction  $u_n(x)$  corresponding to  $\lambda_n$  is called the nodal set and  $l_j^n = x_{j+1}^n - x_j^n$  is defined as the nodal

length of  $u_n(x)$ . Using the nodal datas, some uniqueness results, reconstruction and stability of potential functions have been solved by many authors [18], [23].

Consider  $p$ -Laplacian eigenvalue problem

$$-\left(u'^{(p-1)}\right)' = (p-1)(\lambda^2 - q(x) - 2\lambda r(x))u^{(p-1)} \quad (1.3)$$

with the Dirichlet conditions

$$u(0) = u(1) = 0 \quad (1.4)$$

and Neumann boundary conditions

$$u'(0) = u'(1) = 0 \quad (1.5)$$

where  $p > 1$ ,  $\lambda$  is eigenvalue and  $q(x) \in L^2(0, 1)$ ,  $r(x) \in W_2^1(0, 1)$ .

For  $p = 2$ , equation (2.2) becomes

$$-y'' + [q(x) + 2\lambda r(x)]y = \lambda^2 y. \quad (1.6)$$

This equation is known as diffusion equation or quadratic of differential pencil. The eigenvalue equation (1.6) is of important both classical and quantum mechanics. For example, such problems arise in solving the Klein-Gordon equations, which describe the motion of massless particles such as photons. Sturm-Liouville energy-dependent equations are also used for modelling vibrations of mechanical systems in viscous media (see [12]). We note that in this type problem the spectral parameter  $\lambda$  is related to the energy of the system, and this motivates the terminology “energy-dependent” used for the spectral problem of the form (1.6). Inverse problem of quadratic pencil have been solved by many authors in the references [8], [9], [11], [14], [15], [21], [24], [25], [26].

As in  $p$ -Laplacian Sturm-Liouville problem, for  $p(x) = r(x) = 0$  eigenvalues of the problem (1.3), (1.4) are given

$$\lambda_n = (n\pi_p)^p$$

and associated eigenfunction is denoted by  $S_p(x, \lambda)$ . We will note it by  $S_p(x)$  briefly.

This paper is organized as follows: In section 2, we give asymptotic formula for eigenvalues, nodal points and nodal lengths. In section 3, we give a reconstruction formula for differential pencil by using nodal parameters.

## 2. Asympmtotic Estimates of Nodal Parameters

In this section, we study the properties of eigenvalues of the  $p$ -Laplacian operator (1.3) with the Dirichlet conditions. For this, we must introduce the Prüfer substitution. One can easily obtain similiar results for Neumann problem.

We define a modified Prüfer substitution

$$\begin{aligned} u(x) &= c(x)S_p\left(\lambda^{2/p}\theta(x)\right) \\ u'(x) &= \lambda^{2/p}c(x)S'_p\left(\lambda^{2/p}\theta(x)\right) \end{aligned} \quad (2.1)$$

or

$$\frac{u'(x)}{u(x)} = \lambda^{2/p} \frac{S'_p\left(\lambda^{2/p}\theta(x)\right)}{S_p\left(\lambda^{2/p}\theta(x)\right)}. \quad (2.2)$$

Differentiating the (2.2) with respect to  $x$  and applying Lemma 1.1. one can obtain that

$$\theta'(x) = 1 - \frac{q}{\lambda^2}S_p^p - \frac{2}{\lambda}rS_p^p \quad (2.3)$$

**Theorem 2.1.** The eigenvalues  $\lambda_n$  of the Dirichlet problem (1.3),(1.4) are

$$\lambda_n^{2/p} = n\pi_p + \frac{1}{p(n\pi_p)^{p-1}} \int_0^1 q(t)dt + \frac{2}{p(n\pi_p)^{\frac{p-2}{p}}} \int_0^1 r(t)dt + O\left(\frac{1}{n^{\frac{p+2}{p}}}\right) \quad (2.4)$$

**Proof :** For the problem (1.3),(1.4), Let  $\lambda = \lambda_n, \theta(0) = 0$  and  $\theta(1) = \frac{n\pi_p}{\lambda_n^{2/p}}$ . Firstly, we should integrate both sides of (2.3) on  $[0, 1]$

$$\frac{n\pi_p}{\lambda_n^{2/p}} = 1 - \frac{1}{\lambda_n^2} \int_0^1 q(t)S_p^p(t)dt - \frac{2}{\lambda_n} \int_0^1 r(t)S_p^p(t)dt$$

using the identity

$$\frac{d}{dt} \left[ S_p\left(\lambda_n^{2/p}\theta(t)\right) S'_p\left(\lambda_n^{2/p}\theta(t)\right)^{p-1} \right] = \left(1 - p \left| S_p\left(\lambda_n^{2/p}\theta(t)\right) \right|^p\right) \lambda_n^{2/p}\theta'(t)$$

and Lemma 1.1. (b), we get

$$\begin{aligned} \frac{n\pi_p}{\lambda_n^{2/p}} &= 1 - \frac{1}{\lambda_n^2 p} \int_0^1 q(t)dt - \frac{2}{\lambda_n p} \int_0^1 r(t)dt \\ &\quad + \frac{1}{\lambda_n^2 p} \int_0^1 \frac{q(t)}{\lambda_n^{2/p}\theta'(t)} \frac{d}{dt} \left[ S_p\left(\lambda_n^{2/p}\theta(t)\right) S'_p\left(\lambda_n^{2/p}\theta(t)\right)^{p-1} \right] dt \\ &\quad + \frac{2}{\lambda_n p} \int_0^1 \frac{r(t)}{\lambda_n^{2/p}\theta'(t)} \frac{d}{dt} \left[ S_p\left(\lambda_n^{2/p}\theta(t)\right) S'_p\left(\lambda_n^{2/p}\theta(t)\right)^{p-1} \right] dt. \end{aligned} \quad (2.5)$$

Then, using integration by parts, we have

$$\begin{aligned} \int_0^1 \frac{q(t)}{\lambda_n^{2/p}\theta'(t)} \frac{d}{dt} [S_p S_p^{p-1}] dt &= -\lambda_n^{-2/p} \int_0^1 G\left(\lambda_n^{2/p}\theta(t)\right) \frac{d}{dt} \left( \frac{q(t)}{\theta'(t)} \right) dt \\ &= O\left(\frac{1}{\lambda_n^{2/p}}\right), \end{aligned}$$

where

$$G\left(\lambda_n^{2/p}\theta(x)\right) = S_p\left(\lambda_n^{2/p}\theta(x)\right) S_p'\left(\lambda_n^{2/p}\theta(x)\right)^{p-1}$$

and when  $x = 0, 1$

$$G\left(\lambda_n^{2/p}\theta(x)\right) = 0.$$

Similarly, one can show that

$$\int_0^1 \frac{r(t)}{\lambda_n^{2/p}\theta'(t)} \frac{d}{dt} [S_p S_p'^{p-1}] dt = O\left(\frac{1}{\lambda_n^{2/p}}\right).$$

Inserting these values in (2.5) and after some straightforward computations, we obtain (2.4).

**Theorem 2.2.** For the problem (1.3), (1.4), The nodal points expansion satisfies

$$\begin{aligned} x_j^n &= \frac{j}{n} + \frac{j}{pn^{p+1}(\pi_p)^p} \int_0^1 q(t) dt + \frac{2j}{pn^{\frac{2p-2}{p}}(\pi_p)^{\frac{2p-2}{p}}} \int_0^1 r(t) dt + \frac{2}{(n\pi_p)^{\frac{p}{2}}} \int_0^{x_j^n} r(x) S_p^p dx \\ &\quad + \frac{1}{(n\pi_p)^p} \int_0^{x_j^n} q(x) S_p^p dx + O\left(\frac{j}{n^{\frac{3p+2}{p}}}\right). \end{aligned}$$

**Proof:** Let  $\lambda = \lambda_n$  and integrating (2.3) from 0 to  $x_j^n$ , we have

$$\frac{j \cdot \pi_p}{\lambda_n^{2/p}} = x_j^n - \int_0^{x_j^n} \frac{2r(x)}{\lambda_n} S_p^p dx - \int_0^{x_j^n} \frac{q(x)}{\lambda_n^2} S_p^p dx.$$

By using the estimates of eigenvalues as

$$\frac{1}{\lambda_n^{2/p}} = \frac{1}{n\pi_p} + \frac{1}{p(n\pi_p)^{p+1}} \int_0^1 q(t) dt + \frac{2}{p(n\pi_p)^{\frac{3p-2}{p}}} \int_0^1 r(t) dt + O\left(\frac{1}{n^{\frac{3p+2}{p}}}\right),$$

we obtain

$$\begin{aligned} x_j^n &= \frac{j}{n} + \frac{j}{pn^{p+1}(\pi_p)^p} \int_0^1 q(t) dt + \frac{2j}{pn^{\frac{2p-2}{p}}(\pi_p)^{\frac{2p-2}{p}}} \int_0^1 r(t) dt + \frac{2}{(n\pi_p)^{\frac{p}{2}}} \int_0^{x_j^n} r(x) S_p^p dx \\ &\quad + \frac{1}{(n\pi_p)^p} \int_0^{x_j^n} q(x) S_p^p dx + O\left(\frac{j}{n^{\frac{3p+2}{p}}}\right). \end{aligned}$$

**Theorem 2.3.** As,  $n \rightarrow \infty$ ,

$$l_j^n = \frac{\pi_p}{\lambda_n^{2/p}} + \frac{2}{p\lambda_n} \int_0^1 r(t)dt + \frac{1}{p\lambda_n^2} \int_0^1 q(t)dt + O\left(\frac{1}{\lambda_n^{\frac{4+p}{p}}}\right) \quad (2.6)$$

**Proof :** For large  $n \in \mathbb{N}$ , integrating (2.3) on  $[x_j^n, x_{j+1}^n]$  and then

$$\frac{\pi_p}{\lambda_n^{2/p}} = l_j^n - \frac{2}{\lambda} \int_{x_j^n}^{x_{j+1}^n} r(t)S_p^p dt - \frac{1}{\lambda^2} \int_{x_j^n}^{x_{j+1}^n} q(t)S_p^p dt$$

or

$$\begin{aligned} \frac{\pi_p}{\lambda_n^{2/p}} &= l_j^n - \frac{2}{p\lambda_n} \int_{x_j^n}^{x_{j+1}^n} r(t)dt - \frac{1}{p\lambda_n^2} \int_{x_j^n}^{x_{j+1}^n} q(t)dt + \frac{2}{\lambda_n p} \int_{x_j^n}^{x_{j+1}^n} \frac{1}{\lambda_n^{2/p}\theta'(t)} \frac{d}{dt} [S_p S_p'^{p-1}] r(t)dt + \\ &\quad \frac{1}{\lambda_n^2 p} \int_{x_j^n}^{x_{j+1}^n} \frac{1}{\lambda_n^{2/p}\theta'(t)} \frac{d}{dt} [S_p S_p'^{p-1}] q(t)dt. \end{aligned} \quad (2.7)$$

By Lemma 1.1. and similar process of Theorem 2.1, we obtain that

$$\begin{aligned} \int_{x_j^n}^{x_{j+1}^n} \frac{r(t)}{\lambda_n^{2/p}\theta'(t)} \frac{d}{dt} [S_p S_p'^{p-1}] dt &= - \int_{j\pi_p}^{(j+1)\pi_p} \left( \frac{q(t)}{\lambda_n^{2/p}\theta'(t)} \right)' G(\tau) \frac{d\tau}{\lambda_n^{2/p}\theta'(t)} \\ &= O\left(\frac{1}{\lambda_n^{4/p}}\right), \end{aligned}$$

where  $G(\tau) = S_p(\tau)S_p'(\tau)^{(p-1)}$  and  $\tau = \lambda_n^{2/p}\theta(x)$ . Similarly one can show that

$$\int_{x_j^n}^{x_{j+1}^n} \frac{q(t)}{\lambda_n^{2/p}\theta'(t)} \frac{d}{dt} [S_p S_p'^{p-1}] dt = O\left(\frac{1}{\lambda_n^{4/p}}\right).$$

Inserting this value in (2.7), we obtain

$$l_j^n = \frac{\pi_p}{\lambda_n^{2/p}} + \frac{2}{p\lambda_n} \int_{x_j^n}^{x_{j+1}^n} r(t)dt + \frac{1}{p\lambda_n^2} \int_{x_j^n}^{x_{j+1}^n} q(t)dt + O\left(\frac{1}{\lambda_n^{\frac{4+p}{p}}}\right)$$

and by Theorem 2.1.

$$l_j^n = \frac{1}{n} + \frac{2}{p(n\pi_p)^{p/2}} \int_{x_j^n}^{x_{j+1}^n} r(t)dt + \frac{1}{p(n\pi_p)^p} \int_{x_j^n}^{x_{j+1}^n} q(t)dt + O\left(\frac{1}{n^{\frac{4+p}{p}}}\right).$$

### 3. Reconstruction of Potential Function in Differential Pencil

In this section, we give an explicit formula for potential function. The method used in the proof of the theorem is similar to classical Sturm-Liouville problem [18], [23].

**Theorem 3.1.** Let  $q \in L^2(0, 1)$  and  $r \in W_2^1(0, 1)$ . Then,

$$q(x) = \lim_{n \rightarrow \infty} p\lambda_n^2 \left( \frac{\lambda_n^{2/p} l_j^n}{\pi_p} - \frac{2r(x)}{p\lambda_n} - 1 \right)$$

for  $j = j_n(x) = \max\{j : x_j^n \leq x\}$ .

**Proof :** Applying the mean value theorem for integral to (2.6), with fixed  $n$ , there exists  $z \in (x_j^n, x_{j+1}^n)$ , we obtain

$$l_j^n = \frac{\pi_p}{\lambda_n^{2/p}} + \frac{2}{p\lambda_n} r(z) l_j^n + \frac{1}{p\lambda_n^2} q(z) l_j^n + O\left(\frac{1}{\lambda_n^{\frac{4+p}{p}}}\right)$$

or

$$q(z) = p\lambda_n^2 \left( \frac{\pi_p}{\lambda_n^{2/p} l_j^n} \right) \left( \frac{\lambda_n^{2/p} l_j^n}{\pi_p} - \frac{2r(z)}{p\lambda_n} \frac{\lambda_n^{2/p} l_j^n}{\pi_p} - 1 \right).$$

Considering (2.6), we can write that for  $n \rightarrow \infty$

$$\frac{\lambda_n^{2/p} l_j^n}{\pi_p} = 1,$$

Then,

$$q(x) = \lim_{n \rightarrow \infty} p\lambda_n^2 \left( \frac{\lambda_n^{2/p} l_j^n}{\pi_p} - \frac{2r}{p\lambda_n} - 1 \right).$$

This completes the proof.

**Conclusion 2.4.** In the Theorem 2.1., Theorem 2.2., Theorem 2.3 and Theorem 3.1, taking  $r(x) = 0$  we obtain results of Sturm-Liouville problem given in [16].

**Conclusion 2.5.** In the Theorem 2.1., Theorem 2.2., Theorem 2.3, and Theorem 3.1, taking  $p = 2$ , we obtain results of inverse nodal problem for differential pencil.

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